THE CHINESE UNIVERSITY OF HONG KONG
MATH4010 Suggested solutions to homework 4

If you find any mistakes or typos, please report them to ypyang@math.cuhk.edu.hk

7.11. Solution. Let (x_n) be a sequence in an inner product space. Show that the conditions $\|x_n\| \to \|x\|$ and $\langle x_n, x\rangle \to \langle x, x\rangle$ imply $x_n \to x$.

Proof. Notice that

$$\langle x, x_n\rangle = \overline{\langle x_n, x\rangle} \to \overline{\langle x, x\rangle} = \langle x, x\rangle$$

and hence

$$\begin{aligned}\|x_n - x\|^2 = \langle x_n - x, x_n - x\rangle &= \|x_n\|^2 - \langle x_n, x\rangle - \langle x, x_n\rangle + \|x\|^2 \\ &\to \|x\|^2 - \langle x, x\rangle - \langle x, x\rangle + \|x\|^2 = 0.\end{aligned}$$

It follows that $x_n \to x$.

7.15. Prove that in an inner product space, $x \perp y$ if and only if

$$\|x + \lambda y\| = \|x - \lambda y\|,$$

for all scalars $\lambda \in \mathbb{F}$.

Proof.

$$\begin{aligned}&\|x + \lambda y\| = \|x - \lambda y\|, \quad \forall \lambda \in \mathbb{F} \\ \Longleftrightarrow&\|x + \lambda y\|^2 = \|x - \lambda y\|^2, \quad \forall \lambda \in \mathbb{F} \\ \Longleftrightarrow&\langle x + \lambda y, x + \lambda y\rangle = \langle x - \lambda y, x - \lambda y\rangle, \quad \forall \lambda \in \mathbb{F} \\ \Longleftrightarrow&\|x\|^2 + \lambda\langle y, x\rangle + \bar{\lambda}\langle x, y\rangle + \lambda^2\|y\|^2 = \|x\|^2 - \lambda\langle y, x\rangle - \bar{\lambda}\langle x, y\rangle + \lambda^2\|y\|^2, \quad \forall \lambda \in \mathbb{F} \\ \Longleftrightarrow&\lambda\langle y, x\rangle + \bar{\lambda}\langle x, y\rangle = 0, \quad \forall \lambda \in \mathbb{F}\end{aligned}$$

If $x \perp y$, then $\langle x, y\rangle = 0$ and it can be seen that $\|x + \lambda y\| = \|x - \lambda y\|$, $\forall \lambda \in \mathbb{F}$.

If $\|x + \lambda y\| = \|x - \lambda y\|$, $\forall \lambda \in \mathbb{F}$, then we take $\lambda = \langle x, y\rangle$ to get

$$2|\langle x, y\rangle|^2 = 0 \Longrightarrow \langle x, y\rangle = 0.$$